Department of Mathematics, IIT Bombay

Screening Test for PhD Admissions (2 Dec, 2015)

Time allowed: 2 hours and 30 minutes **Maximum Marks: 40**

Note: Q.1-12 carry 3 marks each. Q.13 carries 4 marks.

Q.1 Let A be a 5×5 matrix s.t. $A^2 = 0$. Compute the least upper bound for its rank.

Q.2 Gram-Schmidt process is applied to the *ordered* basis $\{\mathbf{i}+\mathbf{j}+\mathbf{k}, \mathbf{i}+\mathbf{j}, \mathbf{i}\}$ in $\mathbb{R}^3$. Find the resulting orthonormal basis.

Q.3 Let $A = [a_{ij}]$ be a square matrix of order n whose entries are given as follows. For $1 \leq i, j \leq n$ we have

$$a_{ij} = \begin{cases} ij & \text{if } i \neq j, \\ 1 + ij & \text{if } i = j. \end{cases}$$

Evaluate the determinant of A.

Q.4 Arrange the following matrices with their ranks in a non-decreasing order.

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 2^2 & 2^3 \\ 2^4 & 2^5 & 2^6 \\ 2^7 & 2^8 & 2^9 \end{bmatrix}, \quad R = \begin{bmatrix} 1^2 & 2^2 & 3^2 \\ 4^2 & 5^2 & 6^2 \\ 7^2 & 8^2 & 9^2 \end{bmatrix}.$$

Q.5* If $a_1 \geq 0$, $a_2 \geq 0$ and $a_{n+2} = \sqrt{a_n a_{n+1}}$, find the limit of the sequence $\{a_n\}$.

<u>Equivalent but more humane:</u>

If $a_1 = a$, $a_2 = b$ and $a_{n+2} = \frac{a_n + a_{n+1}}{2}$, find the limit of the sequence $\{a_n\}$. Ans.$=\frac{a+2b}{3}$

Q.6* Let $\{P_n\}$ be a sequence of polynomials such that and for $n = 0, 1, 2, ...$

$$P_0 = 0 \text{ and } P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Assuming the fact that $\{P_n\}$ is convergent point-wise, find the limit function $\lim_{n\to\infty} P_n(x)$.

<u>Equivalent but more humane:</u>

Given $a \in \mathbb{R}$ and

$$a_0 = 0 \text{ and } a_{n+1} = a_n + \frac{a^2 - a_n^2}{2},$$

Find optimal upper and lower bounds for $\{a_n\}$. [Hint: Use induction on n.]

Q.7 Find the values of x, $(x \in \mathbb{R})$ for which the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

is convergent.

Q.8 Find the range of values of α for which $\displaystyle\int_0^{\infty} \frac{\sin x}{x^{\alpha}}$ is convergent (i.e. is finite).

Q.9 A point X is picked uniformly at random from the perimeter of a unit circle. Find the expected value of $|X|$.

Q.10 Suppose the distribution of Y, conditional on $X = x$, is Normal(x, x^2) and the marginal distribution of X is Uniform$(0, 1)$. Find the $Cov(X, Y)$.

Q.11 Let X be an observation from the probability density function

$$f(x|\theta) = \left(\frac{\theta}{2}\right)^{|x|} (1 - \theta)^{1-|x|}, \quad x = -1, 0, 1; \quad 0 \leq \theta \leq 1.$$

Find the maximum likelihood estimator (mle) of θ and its expectation.

Q.12 Let $U_1, U_2, \ldots, U_n$ be i.i.d. Uniform$(0, \theta)$, $\theta > 0$, random variables. Find the uniformly minimum variance unbiased estimator (UMVUE) of $\cos\theta$.

Q.13 Let $A = \begin{bmatrix} 1 & r & r \\ r & 1 & r \\ r & r & 1 \end{bmatrix}$. Find the range of values of r so that A is positive definite.